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Stratification of proper real analytic mappings

and

structural stability of a family of real analytic sets.

FUKUDA Takuo

0. Introduction.

Abstractly what we call the problem of stability can be expressed as follows: We consider an equivalence relation $\sim$ between objects (or morphisms) of some category. Let $\{E_a\}$, $a \in A$, be a family of objects (or morphisms) of this category, the parameter space A being a topological space. We say that E_a is stable in the family $\{E_a\}$ with respect to the relation $\sim$ if there exists a neighborhood U of a in A such that for any point $c \in U$, we have $E_c \sim E_a$. The set K of points $b \in A$ such that E_b is not stable is called the bifurcation set of the family $\{E_a\}$. Then the problem of structural stability is : Is the bifurcation set K nowhere dense in A ?

Now, we consider the following situation: Let A and B be complex or real anlytic spaces or analytic sets. Consider a system of analytic equations

$$F_j(x,y) = 0 \qquad , x \in A, \quad y \in B$$

Then we have a family $\{E_a\}$, $a \in A$, of real analytic sets defined by

$$E_a = \{y \in B | \; F_j(a,y) = 0 \} \qquad .$$

We discuss in the present paper the structural stability of this family in the topological sense : E_a is topologically stable if there is a neighborhood U of a in A such that for any point $c \in U$, E_c is homeomorphic to E_a. Furthermore we are interested in the topological structure of the bifurcation set K of this family.

If we replace the term "analytic" by "algebraic" in the above situation, it is known that the bifurcation set K is constructible (or semi-algebraic) subset of A and nowhere dense in A. So , in this case, we obtain a positive response for the topological stability problem. The main technique is to stratify the projection map $p:G \to A$, where $G =\{ (x,y) \in A \times B | \; F_j(x,y) = 0 \}$ and P is defined by

$$p(x,y) = x.$$

For a method of stratification of $p:G \to A$, see [1].

If A, B and F_j are complex analytic, it is also known that if $p:G \to A$ is proper , that is, if the counter image of a compact set is compact, the fact mentioned above holds exact: K is an analytic subset of A and nowhere dense in A. If $p:G \to A$ is not proper, we can not say any thing about the bifurcation sets K.

In real analytic case, Thom says [6], "une caractérisation intrinsèque de ces ensembles très vraisemblement stratifiés, n'a pas encore été explicitée dans la littérature." The purpose of the present paper is ~~to~~ give a similar response:

THEOREM 1. In real analytic case, if $p:G \to A$ is proper, then the bifurcation set K is a subanalytic subset of A and nowhere dense in A.

The main tools of the proof are the notion of subanalytic sets obtained by H.Hironaka [9] and the stratification of proper real analytic morphisms:

THEOREM 2. A proper real analytic morphism is a stratified map.

Next, we consider the following situation: Let A and B be real analytic or suanalytic sets. and let $f:A \times B \to R$ be a real analytic function. Suppose that B is compact. Then we have a family $\{ f_a \}$, $a \in A$, of real analytic functions defined by

$$f_a : B \to R \quad : \quad f_a(b) = f(a,b).$$

We consider the structural stability of this family in the following sense: f_a is <u>topologically stable</u> if there exists a neighborhood U of a in A such that for any point $c \in U$, f_c is <u>topologically equivalent</u> to f_a: i.e. there exist homeomorphisms $h_1 : B \to B$ and $h_2 : R \to R$ such that the followimg diagramm commutes

$$\begin{array}{ccc} B & \xrightarrow{h_1} & B \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_c} \\ R & \xrightarrow{h_2} & R \end{array}$$

Then we have

THEOREM 3. <u>In the abve situation, the bifurcation set</u> K <u>is suanalytic and nowhere dense in</u> A.

This theorem follows from Thom's second isotopy lemma (proposition 2.5.) and the following theorem.

THEOREM 4. Let A, B and f be as above. Let $F: A \times B \to A \times R$ be the map defined by $F(a,b) = (a, f(a,b))$. Then A admits a Whitney stratification $S(A)$ such that for any stratum X of $S(A)$, the map $F|X \times B: X \times B \to X \times A$ is a Thom mapping over the projection map $p: X \times A \to X$.

In the present paper, we only give the proof of Theorem 1 and theorem 2. For the proof of theorem 3 and 4 see [2].

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1. Whitney stratification.

In which we introduce the notion of stratifications which is due to H.Whitney [7], [8]. Here we recall only the definitions and some properties which we need. For the proof of these properties and more details, we are refered to R.Thom [5] and J.Mather [4].

Let X and Y be differentiable submanifolds of R^n. Let y be a point of Y and let $r = \dim X$. In what follows, $T_p(M)$ denotes the tangent space to a manifold M at a point p of M.

DEFINITION 1.1. We say that the pair (X,Y) satisfies condition (a) at $y \varepsilon Y$ if the following holds: Given any sequence x_i of points in X such that $x_i \to y$ and the tangent space $T_{x_i}(X)$ converges to some r-plane τ, we have $T_y(Y) \subset \tau$.

Here and in what follows " convergence" means convergence in the standard topology on the Grassmannian manifold of r-panes in R^n.

For any two distinct points x,y R^n, the secant $\widehat{xy}$ denotes the line in R^n which is parallel to the line joining x and y and passes through the origin.

Let X, Y be smooth submanifolds of R^n. Let $y \in Y$. Let $r = \dim X$.

DEFINITION 1.2. We say that the pair (X,Y) satisfies condition (b) at y if the following holds. Given any sequences x_i of points in X and y_i of points in Y such that $x_i \neq y_i$, $x_i \to y$ and $y_i \to y$ and such that $T_{x_i}(X)$ converges to some r-plane τ and the secants $\widehat{x_i y_i}$ converge to some line ℓ R^n, we have $\ell \subset \tau$.

We say the pair (X,Y) satisfies condition (a) (resp. (b)) if it satisfies condition (a) (resp. (b)) at every point of Y.

REMARK. (Mather [4].) If (X,Y) satisfies condition (b) at y, then it satisfies condition (a) at y.

DEFINITION 1.3. A W-complex is a set $\mathcal{S} = \{X_\alpha\}$ of connected smooth manifolds in R^n, called strata of $\mathcal{S}$, satisfying the following conditions:

(i) The strata X_α are pair-wise disjoint.

(ii) (X,Y) satisfies condition (b) for any pair (X,Y) of strata of $\mathcal{S} = \{X_\alpha\}$.

(iii) The family $\mathcal{S} = \{X_\alpha\}$ is locally finite: each point of R^n has a neighborhood which meets at most finitely many strata.

DEFINITION 1.4. A stratified set is a subset E of R^n provided a W-complex $S(E) = \{X_\alpha\}$ with $E = X_\alpha$. We call $S(E)$ a Whitney stratification of E.

REMARK. (Mather [4].). The local finitness of strata and the condition (b) imply the condition of frontier: For each stratum X of $S(E)$, its frontier $(\bar{X} - X) \cap E$ is a union of strata.

NOTATION. Let X, Y be two strata of $S(E)$ with $Y \cap \bar{X} \neq \emptyset$. Then by the above remark, we have $Y \subset \bar{X} - X$. We represent this situation by the symbol $Y < X$ and we say Y is incident to X.

2. Stratified mappings and Thom's isotopy lemmas.

Let $E \subset R^n$ and $F \subset R^m$.

DEFINITION. We say that a continuous mapping $f: E \to F$ is a W-morphism or a stratified mapping if there exist stratifications $S(E)$ of E and $S(F)$ of F and the following conditions hold:

(i) f is extendable to a differentiable mapping of a neighborhood of E into R^m.

(ii) For any stratum X of $S(E)$, the image $f(X)$ is contained in a stratum Y of $S(F)$ and the restricted mapping $f|X: X \to Y$ is a submersion.

A W-morphism $f: E \to F$ will be said to be exact if for any stratum X of $S(E)$, $f(X)$ is a stratum of $S(F)$.

REMARK. A proper W-morphism is an exact W-morphism. (See Mather's existence theorem for tubular neighborhoods [4].).

PROPOSITION 2.2. (Thom's first isotopy lemma). If $f:E \to F$ is a proper stratified mapping, then for each stratum Y of $S(F)$, the restricted mapping $f|f^{-1}(Y):f^{-1}(Y) \to Y$ is a locally trivial fibre bundle.

For the proof, see Mather [4], Thom [5] or Fukuda [1] .

DEFINITION 2.3. (Thom's condition a_f.). Let X and Y be smooth submanifolds of R^n and let N be a smooth manifold. Let $f:U \to N$ be a differentiable mapping defined on a neighborhood U of $X \cup Y$ in R^n. Suppose that $f|X$ and $f|Y$ are of constant rank. Then we say the pair (X,Y) satisfies condition a_f at a point $y \in Y$ if the following holds: Given any sequence x_i of points in X converging to y such that the sequence of planes $\ker(f|X)_{x_i}$ converges to a plane τ in the appropriate Grassmannian manifold, we have

$$\ker(f|Y)_y \subset \tau \quad ,$$

where $\ker(f|X)_x$ denotes the kernel of the differential

$$(df|X)_x:T_x(X) \to T_{f(x)}(N)$$

of $f|X:X \to N$.

We say that the pair (X,Y) satisfies condition a_f if it satisfies condition a_f at every point of Y.

DEFINITION 2.4. (Thom mapping.). Let $f:E \to F$ and $g:F \to V$ be stratified mappings. Suppose that V is a connected smooth manifold and it is considered as a stratified set with its trivial stratification $S(V) = \{V\}$. Then we say that f is a Thom mapping over g if for each point p of V and for any pair (X,Y) of strata of $S(E)$, the pair $(X \cap (g\,f)^{-1}(p),\ Y \cap (g\,f)^{-1}(p))$ satisfies condition a_f.

Let $f:E \to F$ be a Thom mapping over $g:F \to V$. For a point p of V, set $E_p = (g\,f)^{-1}(p)$ and $F_p = g^{-1}(p)$.

PROPOSITION 2.5. (Thom's second isotopy lemma.). Let $f:E \to F$ be a proper Thom mapping over a proper stratified mapping $g:F \to V$. Then for any two points p and q of V, the restricted mappings $f|E_p:E_p \to F_p$ and $f|E_q:E_q \to F_q$ are of same topological type: there exist homeomorphisms $h_1:E_p \to E_q$ and $h_2:F_p \to F_q$ such that the following diagramm commutes:

$$\begin{tikzcd} E_p \arrow[r, dashed, "h_1"] \arrow[d, "f"'] & E_q \arrow[d, "f"] \\ F_p \arrow[r, dashed, "h_2"] & F_q \end{tikzcd}$$

For the proof of this proposition, see Mather [4] or Fukuda [1].

3. Subanalytic subsets.

In which we introduce the notion of "subanalicity" that is due to H.Hironaka [3]. All the properties are stated without proof. For the proof, more details or examples, see [3].

DEFINITION 3.1. Let Ω be an open set in R^n. An analytic set $A \subset \Omega$ is a set such that for any point a of Ω, there is a neighborhood U of a in Ω and analytic functions $f_1, \ldots, f_k$ in U such that

$$A \cap U = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0\}.$$

DEFINITION 3.2. (Analytic mappings). Let A_i, $i=1,2$, be analytic sets in open sets Ω_i R^{n_i}. A continuous mapping $f: A_1 \to A_2$ is said to be analytic at a point $a \in A_1$ if there exist a neighborhood U of a in Ω_1 and an analytic mapping $F: U \to R^{n_2}$ with

$$F|A_1 \cap U = f|A_1 \cap U.$$

An analytic mapping is, at least in the present paper, a continuous mapping of an analytic set A_1 into another analytic set which is analytic at every point of A_1.

DEFINITION 3.3. (*Subanalytic subsets*). Let $X \subset \Omega$ be an analytic subset of an open set Ω in R^n. *A subanalytic subset* $A \subset X$ is a set such that for any point a of X there exist an open neighborhood U of a in X and a finite system of analytic sets Y_{ij} and proper real analytic mappings $f_{ij}:Y_{ij} \to X$, $1 \leqq i \leqq p$ and $j = 1,2$, such that

$$A \cap U = \cup_{i=1}^{p} (f_{i1}(Y_{i1}) - f_{i2}(Y_{i2})).$$

PROPOSITION 3.4. *Let* A,B,C *be subanalytic subsets of an analyticset* X. *Then so are* $A \cup B$, $A \cap B$ *and* $A-B$.

PROPOSITION 3.5. *Let* $f:X \to Y$ *be a proper real-analytic mapping*.

(i) *If* B *is a subanalytic subset of* Y, *then so is* $f^{-1}(B)$ *in* X.

(ii) *If* A *is a subanalytic subset of* X, *then so is* $f(A)$ *in* Y.

DEFINITION 3.6. Let A be a subanalytic subset of $X \quad \Omega \quad R^n$. A point $a \in A$ is called *a regular point of* A *of dimension* k if there is a neighborhood U of a, $U \quad \Omega$, such that $A \cap U$ is an analytic submanifold of dimension k of U. A point $a \in A$ is called *singular* if it is not regular.

PROPOSITION 3.7. Let A be a subanalytic subset of an analytic set X. Then we have:

(i) The closure $\bar{A}$ of A in X is subanalytic in X.

(ii) Every connected component of A is subanalytic in X and A has locally finite connectedness in X, ie., every point of X has a neighborhood which meets only a finite number of connected components of A.

(iii) The set of singular points of A is subanalytic in X. The set of regular points of A of dimension p is subanalytic in X.

(iv) Regular points are dense in A.

DEFINITION 3.8. Thanks to the proposition 3.7 (iv), we can define, as usually, the local dimension of a subanalytic set A at a point $a \in A$. And so we can define the dimension of A as the max. of the local dimensions of A.

NOTATION 3.9. Let X and Y be real analytic submanifolds of R^n. $S_b(X,Y)$ will denote the set of points $y \in Y$ such that the pair (X,Y) does not satisfy condition (b) at y.

PROPOSITION 3.10. Let X and Y be real analitic submanifolds of R^n. Assume that $X \cap Y = \emptyset$ and $\bar{X} \supset Y$ and that X and Y are both subanalytic in an openset of R^n. Then there exists a subanalytic subset B of Y such that

(i) B is closed in Y and $\dim B < \dim Y$.

(ii) $B \supset S_b(X,Y)$.

4. Stratification of a subanalytic subset.

In which we give a proof of Hironaka's following theorem:

PROPOSITION 4.1. (Hironaka [3]). Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset R^n$. Then A admits a Whitney stratification whose strata are subanalytic in X.

DEFINITION 4.2. We say that a W-complex $\mathcal{S} = \{Y_\alpha\}$ in R^n is compatible with a submanifold X of R^n if for any stratum Y of $\mathcal{S}$ we have $S_b(X,Y) = \emptyset$.

It is clear that in order to prove the proposition 4.1, it is sufficient to prove the following:

PROPOSITION 4.3. Let A be a subanalytic subset of an analytic set $X \subset \Omega \subset R^n$. Let $X_1, \dots, X_k$ be submanifolds of R^n which are subanalytic in X. Assume that $A \cap X_i = \emptyset$ for each i. Then A admits a Whitney stratification which is compatible with $X_1, \dots, X_k$ and such that each stratum is subanalytic in X.

Proof. We prove the proposition by induction on dimension of A. If $\dim A = 0$, then the proposition is evident. So we assume the proposition holds for every subanalytic set A with $\dim A < m$ and we shall prove it for a subanalytic set A with $\dim A = m$.

Let A_{sp} denote the set of the regular points of A of dimension m. Then by proposition 3.7, A_{sp} and $A-A_{sp}$ are both subanalytic in X and we have $\dim (A-A_{sp}) < \dim A = m$. Since A_{sp} is subanalytic in X and a submanifold of R^n and since $A_{sp} \cap X_i \subset A \cap X_i = \emptyset$, there exists, by proposition 3.10, a subanalytic suset B of A_{sp} such that

(i) B is closed in A_{sp} and $\dim B < \dim A_{sp} = m$.

(ii) $B \supset S_b(X_i, A_{sp})$ for each $i=1,\dots,k$.

Set $C = B \cup (A-A_{sp})$, $A^o = A_{sp} - C$ and set $S(A^o)$ = the set of the connected components of A^o. By proposition 3.7 (ii), $S(A^o)$ is locally finite in Ω, hence $S(A^0)$ is a W-complex which is compatible with $X_1, \dots, X_k$ and such that every stratum is subanalytic in X and disjoint with C.

Since $\dim C < m$, by the hypothesis of our induction, C admits a Whitney stratification $S(C)$ which is compatible with $X_1, \dots, X_k$ and with all of strata of $S(A^o)$.

Thus we have a Whitney stratification $S(A) = S(A^o) \cup S(C)$ that is wanted. Q.E.D.

5. Stratification of a proper real analytic mapping.

In which we prove the following

THEOREM 5.1. Let $f: X \to Y$ be a proper real analytic mapping of a real anlytic set X into another one Y. Let $A \subset X$ and $B \subset Y$ be subanalytic subsets. Suppose that $f(A) \subset B$ and that $f|A: A \to B$ is proper. Then $f|A: A \to B$ is a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B such that any stratum of $S(A)$ (resp. of $S(B)$) is subanalytic in X (resp. in Y).

DEFINITION 5.2. Let $A \subset R^n$ and $B \subset R^m$ and let X (resp. Y) be a submanifolds of R^n (resp. of R^m). Let $f: A \to B$ be a stratified mapping with stratifications $S(A)$ of A and $S(B)$ of B. Then we say that the stratified mapping $f: A \to B$ is caompatible with X (resp. with Y) if so is $S(A)$ (resp. $S(B)$).

The theorem 5.1. is a immediate consequence of the following proposition.

PROPOSITION 5.3. Let $f:X \to Y$ and $A \subset X \subset R^n$, $B \subset Y \subset R^m$ be as in theorem 5.1. Let $X_1,\dots,X_k$ (resp. $Y_1,\dots,Y_\ell$) be submanifolds of R^n (resp. of R^m) which are subanalytic in X (resp. in Y). Assume that $A \cap X_i = B \cap Y_j = \emptyset$ for each i and j. Then there exist Whitney stratifications $S(A)$ of A and $S(B)$ of B such that

(i) $f|A:A \to B$ is a stratified mapping provided $S(A)$ and $S(B)$ and it is compatible with $X_1,\dots,X_k$ and $Y_1,\dots,Y_\ell$.

(ii) The strata of $S(A)$ (resp. of $S(B)$) are subanalytic in X (resp. in Y).

To prove the proposition, we need

LEMMA 5.4. (Bertini-Sard). Let $f:X \to Y$ be a proper real analytic map, where both X and Y are smooth. Then there exist a subset S of Y such that

(i) S is closed and subanalytic in Y and $\dim S < \dim Y$.

(ii) for every connected component U of $Y-S$, either $f^{-1}(U) = \emptyset$ or f induces a submersion from $f^{-1}(U)$ to U.

For the proof see [3].

PROOF OF PROPOSITION 5.3.

We prove the proposition by induction on dim B. The verificatòon for the case dim B = O is immediate fro proposition 4.3. So we assume that the proposition holds for subanalytic set B with dim $B^{<}$ p and we shall prove it for a subanalytic set B of dimension p.

Let B_{sp} denote the set of the regular points of B of dimension p. By proposition 3.10, there is a coled suanalytic subset B_1 of B_{sp} such that dim $B_1 <$ dim B_{sp} and B_1 $S_b(Y_j, B_{sp})$ for each $j=1,\dots,\ell$. Set $B_0 = B_{sp} - B_1$ and $A_0 = A \ f^{-1}(B_{sp})$. Then A_0 is subanalytic in X and so is B_0 in Y. By proposition 4.3 , A_0 admits a Whitney stratification $S(A_0)$ which is compatible with $X_1,\dots,X_k$ and such that each stratum is subanalytic in X.

Now for each stratum of $S(A_0)$, consider the restricted map $f|W: W \to B_0$ and set $\Sigma_W = \{ x \in W \mid$ the rank of $f|W$ at x $< \dim B_{sp} = p \}$. Then Σ_W is subanalytic in X. Since $S(A_0)$ is locally finite, $\Sigma = \cup \Sigma_W$ is subanlytic in X and closed in A_0.

Then $f(\Sigma)$ and its closure $\overline{f(\Sigma)}$ are subanalytic in Y and dim $\overline{f(\Sigma)}$ dim B. Set $B_{00} = B_0 - \overline{f(\Sigma)}$ and $A_{00} = A \cap f^{-1}(B_{00})$. Set

$$S(A_{00}) = \{ W \cap A_{00} \mid W \in S(A_0) \}$$

$S(B_{00})$ = the set of the connected components of B_{00}.

With these stratifications $S(A_{00})$ and $S(B_{00})$, $f: A_{00} \to B_{00}$ is a stratified mapping which is campatible with $X_1,\dots,X_k$ and $Y_1,\dots,Y_\ell$.

$B - B_{00}$ is closed in B and $\dim (B-B_{00})$ $\dim B = p$. So by the hypothesis of our induction, there exist stratifications $S(B-B_{00})$ and $S(A-A_{00})$ with which $f:A-A_{00} \to (B-B_{00})$ is a stratified mapping such that it is compatible with $X_1,\ldots,X_k,$ $Y_1,\ldots$ $,Y_\ell$ and with all strata of $S(A_{00})$ and $S(B_{00})$.

Thus we have stratifications $S(A) = S(A_{00}) \cup S(A - A_{00})$ and $S(B) = S(B_{00}) \cup S(B - B_{00})$ which satisfy the conditions in the proposition. Q.E.D.

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